

# Local indistinguishability: more nonlocality with less entanglement

Michał Horodecki, Aditi Sen(De), Ujjwal Sen, and Karol Horodecki

*Institute of Theoretical Physics and Astrophysics, University of Gdańsk, 80-952 Gdańsk, Poland*

We provide a first operational method for checking indistinguishability of orthogonal states by local operations and classical communication (LOCC). This method originates from the one introduced by Ghosh *et al.* (Phys. Rev. Lett. **87**, 5807 (2001)), though we deal with *pure* states. We apply our method to show that an arbitrary complete multipartite orthogonal basis is indistinguishable by LOCC, if it contains at least one entangled state. We also show that probabilistic local distinguishing is possible for full basis *if and only if* all vectors are product. We employ our method to prove local indistinguishability in an example with sets of pure states of  $3 \otimes 3$ , which shows that one can have *more nonlocality with less entanglement*, where “more nonlocality” is in the sense of “increased local indistinguishability of orthogonal states”. This example also provides, to our knowledge, the only known example where  $d \otimes d$  are locally indistinguishable.

Orthogonal quantum state vectors can always be distinguished if there are no restrictions to measurements that one can perform. If the vectors are states of a system consisting of two distant subsystems, then there can be natural restrictions for the measurements that can be done. In particular, if Alice and Bob (the parties holding the subsystems) cannot communicate quantum information, their possibilities significantly decrease. Intuitively one feels that in such a case, there will be a problem with distinguishing orthogonal entangled states, while product ones should remain distinguishable. The first result in this area was rather surprising: in Ref. [1] the authors exhibited a set of orthogonal bipartite pure *product* states, that *cannot* be distinguished with certainty by local operations and classical communication (LOCC) [2]. Another counterintuitive result was obtained in Ref. [3]: *any* two orthogonal multipartite states *can* be distinguished from each other by LOCC, irrespective of how entangled they are. The latter result was greatly extended in Refs. [4]. There is therefore a general question: which sets of orthogonal states are locally distinguishable?

To find that a given set is *distinguishable* [5], one usually needs to build a suitable protocol. To show that the states are *not* distinguishable, one can try to eliminate all possible measurements as in [6]. Another way is to employ somehow the theory of entanglement [2, 7, 8, 9]. A typical statement proving such indistinguishability would be then: Alice and Bob cannot distinguish the states, as they would increase entanglement otherwise (which is impossible by LOCC). The advantage of the latter method is that it allows to estimate the entanglement resources needed to distinguish the states, that are non-distinguishable by LOCC.

In Ref. [10], this approach was first used to check distinguishability between two mixed states (we will call it TDL method). Another powerful method based on entanglement was recently designed in Ref. [11] (we will call it GKRS method). In this paper, building on those two concepts, we introduce first method that is *operational*, i.e. it allows for systematic numerical checks. Moreover the method allows to obtain powerful analytic

results. Our approach provides a strong tool for investigation of distinguishability of sets of bipartite pure states, because it bases on deciding whether some pure state can be transformed into some other pure states by LOCC, the latter issue being completely solved in a series of papers on entanglement measures and entanglement manipulations with pure states [8, 12, 13, 14]. Using it, we show that any full basis of an arbitrary number of systems is not distinguishable, if at least one of the vectors is entangled [15]. For  $2 \otimes n$  systems it is then also “only if”, as product bases are distinguishable in this case [16]. The result applies also to probabilistic distinguishability: we obtain that a full basis is probabilistically distinguishable *if and only if* all vectors are product. As an illustration of the effectiveness of our presented method, we consider an example of local indistinguishability of an *incomplete* basis which exhibits that it is possible to obtain *more nonlocality with less entanglement*. To our knowledge, this is also the only known example of  $d \otimes d$  indistinguishable states in  $d \otimes d$ .

The application of entanglement theory to the problem of local distinguishability is not immediate. Imagine, that we want to distinguish between the four Bell states [18]. If we were able to apply by LOCC just the von Neumann measurement, then we could obviously create entanglement. Namely, if Alice and Bob start with any initial state (hence also possibly a disentangled one), after the von Neumann measurement, it collapses into one of Bell states. This is of course impossible. We cannot however conclude at this moment, that they are indistinguishable. The clue is that we could distinguish between them, while destroying them during the process. Thus Alice and Bob would get to know what state they shared, but the potential entanglement would be destroyed. This is actually the case in the Walgate *et al.* protocol [3], where one distinguishes between any two orthogonal (possibly) entangled states.

To employ entanglement theory in the distinguishability question, a more clever method should be applied. The general hint is to apply the measurement to some larger system. This concept is a basis for the TDL and

GKRSS methods. In the first one [10] the authors considered a state of four systems A, B, C, D:  $\psi = \psi_{AB} \otimes \psi_{CD}$  where  $\psi_{AB}$  and  $\psi_{CD}$  are maximally entangled states. Then the measurement is applied to the AB part (cf. [17]). If the state after measurement is entangled, then one concludes that the measurement cannot be done by use of LOCC, because entanglement can not be produced between the AB part and the CD part, without interaction between the two parts.

The GKRSS method [11] is the following. Given the set of orthogonal states  $\{\psi_i^{AB}\}_{i=1}^k$  to be distinguished, one builds a mixed state

$$\varrho = \sum_i p_i |\psi_i\rangle \langle \psi_i| \otimes |\phi_i\rangle \langle \phi_i| \quad (1)$$

where  $\phi_i$  are some entangled states of the CD system. If Alice(A) and Bob(B) are able to distinguish between the states  $\psi_i$  they can tell the result of their measurement to Claire(C) and Danny(D), who will then share states  $\phi_i$  with probability  $p_i$ . One now compares the initial entanglement  $E(\varrho)$  measured across the AC:BD cut and the final one given by  $\sum_i p_i E(\phi_i)$  according to any chosen entanglement measure  $E$ . If the states  $\psi_i$  are distinguishable by LOCC, then the final entanglement cannot be greater than the initial one; otherwise one could increase entanglement by LOCC [19]. Thus, if we have

$$E(\varrho) < \sum_i p_i E(\phi_i) \quad (2)$$

then the states  $\psi_i$  are not distinguishable by LOCC. In Refs. [11, 21] distillable entanglement was used as  $E$ .

Let us now exhibit the method of the present Letter. It is a modification of the GKRSS method but an *operational* one. Namely instead of classical correlations between AB and CD we will use quantum correlations. Consequently mixture (1) is replaced by the *superposition*

$$\psi_{ABCD} = \sum_i \sqrt{p_i} |\psi_i^{AB}\rangle |\phi_i^{CD}\rangle \quad (3)$$

The states  $\phi_i$  will be used here essentially to *detect* as to whether a set of states are locally indistinguishable and as such we shall henceforth call them “detectors”. At a first glance it seems that this approach should fail, because the pure state is unlikely to have small entanglement. In [11] where mixtures are used, the possibility for the initial state  $\varrho_{ABCD}$  to be separable in the AC:BD cut was much larger, as mixed states are less coherent than pure ones; for a pure state to be separable, it has to be product, while for mixed states, the very mixedness can decrease entanglement, or even produce separability [22]. Let us however exhibit the following example. Suppose that Alice and Bob are to distinguish between the Bell states  $|B_i\rangle$  [18]. As detectors, we take the same states (as in [11]). Our pure state is thus

$$|\psi_B\rangle_{ABCD} = \frac{1}{2} \sum_{i=1}^4 |B_i\rangle_{AB} |B_i\rangle_{CD} \quad (4)$$

One can see that this state can be written as

$$\frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)_{AC} \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)_{BD} \quad (5)$$

So it turns out that it is *product* in AC:BD cut, so that our method will work. Assuming now the four Bell states to be locally distinguishable would immediately imply that the state  $|\psi\rangle$  is entangled in the AC:BD cut which is the desired contradiction. This result was obtained in [11] and their mixed state  $\varrho_{ABCD} = 1/4 \sum_i |B_i\rangle \langle B_i| \otimes |B_i\rangle \langle B_i|$  turned out to be separable in AC:BD (see also [23]). Here we have a pure state which is product. Note that in this particular example, our method, even though originating from the GKRSS approach, coincides with the TDL method.

The advantage of our approach over the GKRSS method is that for mixed states, it is usually hard to check the relation (2) for different entanglement measures. In our case we have pure states on both sides of the inequality, for which the set of all needed measures is known [8, 13]. Even more: Jonathan and Plenio [14], generalizing the Nielsen result [12], have obtained a necessary and sufficient condition for the transformation from a pure state  $\phi$  to an ensemble of pure states  $\{p_i, \phi_i\}$ . The condition is efficiently computable. Namely, let  $\lambda$  and  $\lambda_i$  be vectors of the Schmidt coefficients of  $\phi$  and  $\phi_i$  respectively. Then the LOCC transition  $\phi \rightarrow \{p_i, \phi_i\}$  is possible if and only if the vector  $\sum_i p_i \lambda_i$  majorizes  $\lambda$  [24]. So our method consists of the following steps

- (1) Given the states  $\{\psi_i^{AB}\}_{i=1}^k$  to be distinguished, choose  $k$  detectors  $\phi_i^{CD}$  and probabilities  $p_i$ .
- (2) Applying Jonathan-Plenio criterion [14], check if the transition  $\psi_{ABCD} \rightarrow \{p_i, \phi_i^{CD}\}$  is possible by LOCC (in AC:BD) where  $\psi_{ABCD}$  is of the form (3).

If the transition is impossible, the set of orthogonal states  $\{\psi_i\}_{i=1}^k$  are indistinguishable by LOCC. The item (1) can be formulated more generally in the following way: (1a) Choose  $\psi_{ABCD}$  such that its reduction  $\varrho_{AB}$  has the support spanned by  $\psi_i^{AB}$ 's; (1b) Determine detectors  $\phi_i^{CD}$  by writing  $\psi_{ABCD}$  by means of  $\psi_i^{AB}$ . Let us mention here that we do not know of any example of a set of locally indistinguishable orthogonal states whose local indistinguishability is in principle not obtainable by our method.

Now we will apply our method to obtain the following proposition, where in fact we do not need an explicit use of the Jonathan-Plenio criterion.

**Proposition.** Let  $\psi_i^{AB}$  be a full orthogonal basis of an  $m \otimes n$  system. Then we have: (1) If at least one of the vectors is entangled (see [15]), the set cannot be perfectly distinguished by LOCC; (2) The set can be probabilistically distinguished if and only if all vectors are product.

**Remark.** We will not have “if and only if” for item (1) because there are orthogonal product bases that cannot be distinguished [1]. However item (1) would be “only if”

in  $2 \otimes n$ , as all product bases are locally distinguishable there [16]. Note also that item (2)  $\Rightarrow$  item (1).

**Proof.** Consider the four party state  $|\psi\rangle_{ABCD} = (1/\sqrt{m} \sum_{i=1}^m |ii\rangle_{AC})(1/\sqrt{n} \sum_{j=1}^n |jj\rangle_{BD})$  shared between Alice, Bob, Claire and Danny, which is product across the AC:BD cut. Written in AB:CD, this state takes the form  $1/\sqrt{mn} \sum_{k=1}^{mn} |k\rangle_{AB} |k\rangle_{CD}$ .

Let  $\{|\psi_1\rangle, |\psi_2\rangle, \dots, |\psi_{mn}\rangle\}$  be a set of  $mn$  orthonormal states of an  $m \otimes n$  system. We choose a unitary operator  $U$  such that  $U|k\rangle = |\psi_k\rangle$  for all  $k = 1, 2, \dots, mn$ . We now use the  $U \otimes U^*$  invariance of the state  $|\psi\rangle$  in the AB:CD cut (see e.g. [25]) and write it as  $1/\sqrt{mn} \sum_{k=1}^{mn} |\psi_k\rangle_{AB} |\psi_k\rangle_{CD}^*$ , where the complex conjugation is in the computational basis.

Therefore if Alice and Bob are able to locally distinguish between the  $|\psi_k\rangle$ s, they could ring up Claire and Danny to tell which state they share, resulting in the creation of the corresponding correlated state  $|\psi_k\rangle^*$  between Claire and Danny.

Now if at least one among the  $|\psi_k\rangle$ s is entangled, an assumption of local distinguishability of the  $|\psi_k\rangle$ s would imply that the state  $|\psi\rangle$  has a nonzero amount of entanglement in the AC:BD cut [26]. But this is forbidden as  $|\psi\rangle$  is product in the AC:BD cut.

Note that the above reasoning goes through irrespective of whether the local distinguishing protocol for the  $|\psi_k\rangle$ s is deterministic or probabilistic. This proves that an arbitrary complete set of orthogonal states of any bipartite system is locally indistinguishable (deterministically as well as probabilistically) if at least one of the vectors is entangled. (Note that for the desired contradiction, the probabilistic protocol must have nonzero probability for at least one entangled state.)

On the other hand, a given complete product basis  $\{|v_i\rangle\}$  can be distinguished by von Neumann measurement  $\sum_i |v_i\rangle\langle v_i|(\cdot)|v_i\rangle\langle v_i|$ . This is a separable operation [7, 32] of the form  $\sum_i A_i \otimes B_i(\cdot) A_i^\dagger \otimes B_i^\dagger$ . Such an operation can be probabilistically implemented by Alice and Bob [27] (it was first proven in [17]): they pick random  $i$ , and probabilistically perform operation  $A_i \otimes B_i(\cdot) A_i^\dagger \otimes B_i^\dagger$ .  $\square$

**Generalisation of the proposition.** In  $d_1 \otimes d_2 \otimes \dots \otimes d_N$ , a full orthogonal basis can be distinguished probabilistically if and only if all vectors are product (i.e., of the form  $|\eta_1\rangle \otimes |\eta_2\rangle \otimes \dots \otimes |\eta_N\rangle$ ) [28].

The “only if” part of the generalised proposition is immediate, from the Proposition for the bipartite case, once we note that a multipartite entangled state must be entangled in at least one bipartite cut. Note also that if a set of multipartite states is indistinguishable in a bipartite cut, it would obviously remain so, if we lessen the allowed set of operations by restricting the parties within one cut to remain at distant locations. Since multipartite separable maps can be preformed probabilistically (the same reasoning as above), we obtain also the “if” part. Note that our presented method for testing local

indistinguishability of a set of *bipartite* orthogonal states cannot be extended in its full generality to the multipartite situation as the Jonathan-Plenio criterion [14] has not been as yet generalised to more than two parties.

To see the effectiveness of the presented method, we apply it to obtain an interesting example of indistinguishability of an *incomplete* basis of orthogonal states. First, note that the set  $S$  consisting of the following maximally entangled states (without normalisation) in  $3 \otimes 3$  are distinguishable locally:

$$\begin{aligned} \psi_1 &= |00\rangle + \omega |11\rangle + \omega^2 |22\rangle, \psi_2 = |00\rangle + \omega^2 |11\rangle + \omega |22\rangle, \\ \psi_3 &= |01\rangle + |12\rangle + |20\rangle. \end{aligned} \quad (6)$$

( $\omega$  is a nonreal cube root of unity.) The set  $S$  can be distinguished locally by making a projective measurement (on any one side) in the basis  $\{1/\sqrt{3}(|0\rangle + |1\rangle + |2\rangle), 1/\sqrt{3}(|0\rangle + \omega |1\rangle + \omega^2 |2\rangle), 1/\sqrt{3}(|0\rangle + \omega^2 |1\rangle + \omega |2\rangle)\}$  and a subsequent classical communication to the other party (see also [29]).

Having shown this, what would be our expectation for the set of states containing the same states as in  $S$  but for the last state  $|\psi_3\rangle$ , which is replaced by a *product* state  $\psi'_3 = |01\rangle$ ? The above Propositions seem to indicate that as we put more and more entanglement into the system, the system tends to become locally indistinguishable. This is also the expectation obtained from the recent work of Walgate and Hardy [6]. But one can check by taking  $B_i$ s ( $i = 1, 2, 3$ ) as detectors and with probabilities  $p_i$  as (.16, .16, .68), that the transition  $\sum_{i=1}^2 \sqrt{p_i} |\psi_i\rangle_{AB} |B_i\rangle_{CD} + \sqrt{p_3} |\psi'_3\rangle_{AB} |B_3\rangle_{CD} \rightarrow \{p_i, |B_i\rangle_{CD}\}$  is forbidden by the Jonathan-Plenio criterion [14]. Consequently the set  $S'$ , containing the states (without normalisation)

$$\begin{aligned} \psi_1 &= |00\rangle + \omega |11\rangle + \omega^2 |22\rangle, \psi_2 = |00\rangle + \omega^2 |11\rangle + \omega |22\rangle, \\ \psi'_3 &= |01\rangle \end{aligned} \quad (7)$$

is *indistinguishable* by LOCC [30]. This simple example shows that the intuition that we tried to obtain from our Propositions as well as from the work of Walgate and Hardy [6] is not true. Reducing entanglement from the system can in fact *increase* the nonlocality of the system. This may therefore further the process of “disentangling” nonlocality (in the sense of local indistinguishability) from entanglement [1, 3, 4, 16]. Note that, to our knowledge, this is the only known example of a set of  $d$  indistinguishable states in  $d \otimes d$ .

Since our method is based on entanglement measures [19], there is a question, whether all operations that cannot be performed by LOCC would increase at least one entanglement measure. Most likely it is the case, i.e. the set of LOCC doable operations is described by the set of entanglement measures.

To conclude, we provide a powerful method allowing for efficient investigation of indistinguishability of orthog-

onal vectors via LOCC. We were able to prove general statements for indistinguishability of full bases as well as to provide a counterintuitive example. The question arises whether our method gives the if and only if criterion. In other words, given an ensemble, is it true that they are indistinguishable by LOCC if and only if we can find such detectors so that our method will detect indistinguishability of the ensemble? For example, there exist sets of product states that can be distinguished by separable operations [32] but not by LOCC [1, 16, 33]. Can our method detect such cases? If the answer is “yes”, it would imply that there is an entanglement measure that *can* increase under separable operations (even though it of course cannot increase under LOCC). In our method we go from pure states to pure states, and the set of entanglement measures that are responsible for such possibility is well known and finite [8, 14]. They are sums of squares of  $k$  largest Schmidt coefficients ( $k = 1, \dots, d$ , where  $d$  is dimension of subsystem). There remains an open question as to whether they could increase under separable operations. If the answer is “yes”, then our method can be applied to analyse distinguishability of aforementioned product states. It is however clear that we could not then apply our method with the initial state as product with respect to AC:BD cut. This is because separable operations cannot produce entangled state out of product ones, but can distinguish between the states of interest.

We thank C.H. Bennett for a discussion at the European Research Conference on Quantum Information in San Feliu de Guixols, 2002 and L. Hardy for a discussion at the International Conference on Quantum Information in Oviedo, 2002. We also thank S. Ghosh, G. Kar, J. Oppenheim, A. Roy, D. Sarkar, B. Synak and J. Walgate. The work is supported by the European Community under project EQUIP, Contract No. IST-11053-1999 and by the University of Gdańsk, Grant No. BW/5400-5-0236-2.

- 
- [1] C.H. Bennett, D.P. DiVincenzo, C.A. Fuchs, T. Mor, E. Rains, P.W. Shor, J.A. Smolin, and W.K. Wootters, Phys. Rev. A **59**, 1070 (1999).
  - [2] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).
  - [3] J. Walgate, A.J. Short, L. Hardy, and V. Vedral, Phys. Rev. Lett. **85**, 4972 (2000).
  - [4] S. Virmani, M.F. Sacchi, M.B. Plenio, and D. Markham, Phys. Lett. A **288**, 62 (2001); Y.-X. Chen and D. Yang, Phys. Rev. A **64**, 064303 (2001); *ibid.* **65**, 022320 (2002).
  - [5] In this Letter, “distinguishable” (“indistinguishable”) means “locally distinguishable” (“locally indistinguishable”). And we only consider indistinguishability of *orthogonal* states.
  - [6] J. Walgate and L. Hardy, Phys. Rev. Lett. **89**, 147901 (2002).
  - [7] V. Vedral, M.B. Plenio, M.A. Rippin, and P.L. Knight, Phys. Rev. Lett. **78**, 2275 (1997); V. Vedral and M.B. Plenio, Phys. Rev. A **57**, 1619 (1998).
  - [8] G. Vidal, J. Mod. Opt. **47**, 355 (2000).
  - [9] M. Horodecki, Q. Comp. Inf. **1**, 7 (2001).
  - [10] B.M. Terhal, D.P. DiVincenzo, and D.W. Leung, Phys. Rev. Lett. **86**, 5807 (2001).
  - [11] S. Ghosh, G. Kar, A. Roy, A. Sen(De), and U. Sen, Phys. Rev. Lett. **87**, 277902 (2001).
  - [12] M.A. Nielsen, Phys. Rev. Lett. **83**, 436 (1999).
  - [13] G. Vidal, Phys. Rev. Lett. **83**, 1046 (1999).
  - [14] D. Jonathan and M.B. Plenio, Phys. Rev. Lett. **83**, 1455 (1999).
  - [15] Note that it is not possible to have a complete basis in a tensor product Hilbert space where all but *one* states are product [16].
  - [16] C.H. Bennett, D.P. DiVincenzo, T. Mor, P.W. Shor, J.A. Smolin, and B.M. Terhal, Phys. Rev. Lett. **82**, 5385 (1999).
  - [17] J.I. Cirac, W. Dür, B. Kraus, and M. Lewenstein, Phys. Rev. Lett. **86**, 544 (2001).
  - [18] The four Bell states are  $|B_{1,2}\rangle = 1/\sqrt{2}(|00\rangle \pm |11\rangle)$  and  $|B_{3,4}\rangle = 1/\sqrt{2}(|01\rangle \pm |10\rangle)$ .
  - [19] In the Vidal approach [8], entanglement (or entanglement measure) is defined simply as a quantity that does not increase under LOCC (cf. [20]).
  - [20] S. Popescu and D. Rohrlich, Phys. Rev. A **56**, R3319 (1997).
  - [21] S. Ghosh, G. Kar, A. Roy, D. Sarkar, A. Sen(De), and U. Sen, Phys. Rev. A **65**, 062307 (2002).
  - [22] K. Życzkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, Phys. Rev. A **58**, 883 (1998); K. Życzkowski, Phys. Rev. A **60**, 3496 (1999).
  - [23] J.A. Smolin, Phys. Rev. A **63**, 032306 (2001).
  - [24] If  $x = (x_1, \dots, x_d)$  and  $y = (y_1, \dots, y_d)$  are real  $d$  dimensional vectors,  $x \prec y$  ( $x$  is majorized by  $y$ ) if  $\sum_j^k x_j^\downarrow \leq \sum_j^k y_j^\downarrow \quad \forall k \in \{1, \dots, d\}$ , where  $x_j^\downarrow$  and  $y_j^\downarrow$  are elements of  $x$  and  $y$ , set in decreasing order.
  - [25] M. Horodecki and P. Horodecki, Phys. Rev. A **59**, 4206 (1999).
  - [26] Note that  $|\phi\rangle^*$  is entangled whenever  $|\phi\rangle$  is entangled.
  - [27] There is a direct, geometrical argument showing that a complete product basis is always probabilistically distinguishable [C.H. Bennett, private communication].
  - [28] The entangled state needed for the indistinguishability in the statement of the generalised proposition may not be a genuine  $N$ -party entanglement. E.g., in the 3-qubit case, even a state  $(a|00\rangle + b|11\rangle) \otimes |0\rangle$  in a complete orthogonal basis would be sufficient for local indistinguishability of the basis.
  - [29] Y.-X. Chen and D. Yang, quant-ph/0204152.
  - [30] The three states  $\psi_1$ ,  $\psi_2$  and  $a|01\rangle + b|12\rangle + c|20\rangle$  are indistinguishable for a continuous region around  $(a, b, c) = (1, 0, 0)$  for the same detectors and probabilities. We are willing to *conjecture* that  $\psi_1$ ,  $\psi_2$  and  $a|01\rangle + b|12\rangle + c|20\rangle$  are locally indistinguishable for the whole range of  $(a, b, c)$  except when the last is maximally entangled.
  - [31] Actually in the considered case, all final states have the same  $\lambda_i$ , so that the Jonathan-Plenio criterion reduces to Nielsen’s one [12].
  - [32] E.M. Rains, Phys. Rev. A **60**, 173 (1999).
  - [33] D.P. DiVincenzo, T. Mor, P.W. Shor, J.A. Smolin, and B.M. Terhal, quant-ph/9908070.
  - [34] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).